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RESISTANCE TO THE FREE STEADY-STATE MOTION OF A SPHERE IN A VISCOUS MEDIUM

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An approximate method is proposed for calculating the drag and velocity of spherical particles in a viscous medium.

Numerous experimental investigations [1–3], expressed in the form of Prandtl, Rayleigh, etc. diagrams, have established a graphical relation between drag and Reynolds number. So far, however, a general mathematical relation between these quantities has not been found, and the resistance to the motion of a sphere in a viscous medium is determined by the Stokes, Prandtl-Allen, and Newton-Rittinger laws in the laminar, transition, and turbulent regimes, respectively.

In this paper, an approximate relation between these three laws is proposed on the basis of an analysis of Oseen's theoretical solution.

Oseen's theoretical formula generalizes the laws of Stokes and Newton. This becomes especially obvious if we represent it in the form

$$F = C_V \mu dv + C_D S \frac{\rho v^2}{2}. \quad (1)$$

However, as pointed out in [4], Oseen's formula does not have special advantages over the Stokes formula and is experimentally substantiated in the same range of Reynolds numbers. This is probably because of the assumptions made in determining the constant coefficients in Oseen's theoretical solution (according to Oseen $C_D = 4.5$ [4]; at the same time, according to Newton and Rittinger $C_D = 0.5$ [5]). If we take the value of C_D in Oseen's formula equal to the value proposed by Newton and Rittinger, it is then experimentally confirmed not only in the region of application of the Stokes formula but also in the region of application of the Newton-Rittinger formula. Consequently, Oseen's formula becomes inapplicable only in the transition regime.

This conclusion can also be drawn from an analysis of the boundary conditions and starting assumptions in Oseen's theoretical solution, which have the form

$$\text{at } r = r_0 \quad v_x = v_y = v_z = 0; \text{ as } r \to \infty \quad v_x \to v;\ v_y \to 0;\ v_z \to 0. \quad (2)$$

At points remote from the sphere it is assumed that

$$v_x = v + v'_x;\ v_y = v'_y;\ v_z = v'_z, \quad (3)$$

where v'_x, v'_y, v'_z are quantities small as compared with v_x and may be neglected.

Using boundary conditions in the form (2), Stokes solved the problem of the motion of a fluid at points close to a sphere or, what amounts to the same thing, the motion of a sphere at very small Reynolds numbers [4].

The joint use of boundary conditions in the form of (2) and (3) enabled Oseen to solve the problem of fluid motion not only close to a sphere but also at points remote from it. However, if we keep in mind that v_x varies continuously from zero at the surface to a value v at points remote from the sphere, while v'_y and v'_z vary from zero at the surface of the sphere to zero at infinity, passing through a maximum between these extreme positions, it becomes obvious that there is an intermediate region where it is impossible to neglect v'_x, v'_y, and v'_z. In this region as the Reynolds numbers increase there is a progressive decrease in the frictional contribution to the total resistance of the medium, while the part played by the hydrodynamic drag increases. This is not taken into account in Oseen's solution, and therefore his formula cannot be confirmed over the entire range of Reynolds numbers.

Let us consider the kinematic situation close to a sphere moving in a viscous medium, making a series of assumptions concerning the nature of the distribution of the hydrodynamic pressure forces, which, it seems to us, make it possible to apply Oseen's formula to the transition regime also.

It is known that a sphere moving in a viscous medium drags along with it the adjacent layers of fluid, with which it is connected by molecular bonds and which, like the sphere, experience the dynamic pressure of the medium. Since these layers exert a force on the sphere, in determining the dynamic pressure of the fluid it is necessary to consider the total area S', which is equal to the frontal surface of the sphere and the area of the entrained layers of fluid in the plane of the maximum cross section.

The velocities of the individual fluid particles in the entrained layer are different and depend both on the shape, size, and velocity of the particle and on its distance from the surface of the sphere. Prandtl [4] has shown that the velocity varies from a value equal to the velocity at the surface of the sphere to zero at a distance δ from the surface equal to the thickness of the boundary layer.

The dynamic pressure of the fluid on the boundary layer can be accurately determined in each specific case only if the law of distribution of the velocity of the individual particles within it is known. For the purposes of an approximate determination of the dynamic pressure it is possible to substitute for the fluid layer entrained by the sphere an equivalent layer of reduced thickness $\lambda = k_1\delta$, in which the velocity of the individual particles is constant and equal to the velocity of the sphere, and then verify the correctness of this substitution by experiment. Then, the area S' used in determining the dynamic pressure of the medium is equal to

$$S' = S + S_1 = \frac{\pi d^2}{4} + \pi (d\lambda + \lambda^2), \tag{4}$$

and Oseen's formula reduces to the form

$$F = C_V \mu dv + C_D \frac{\pi d^2}{4} \frac{\rho v^2}{2} + C_D' \pi d\lambda \frac{\rho v^2}{2} + C_D' \pi \lambda^2 \frac{\rho v^2}{2}. \tag{5}$$

It is clear from (5) that the difference from Oseen's formula consists in the two additional terms which are dominant in the transition region.

In accordance with boundary layer theory, the boundary layer formed in flows with appreciable Reynolds numbers has a thickness δ on the order of $d/\sqrt{\mathrm{Re}}$, [4, 5], i. e.,

$$\lambda = k_1 \delta = k_1 k_2 \, d / \sqrt{dv\rho/\mu}. \tag{6}$$

Representing (5) in dimensionless parameters, we obtain

$$\psi = \frac{A}{\mathrm{Re}} + \frac{B}{\sqrt{\mathrm{Re}}} + C_D. \tag{7}$$

Formula (7) generalizes the known laws. The first term corresponds in general form to the Stokes law, the second to the Prandtl-Allen law, the third to Newton's law, and the third and first combined to Oseen's law. Moreover, it resembles the empirical three-term formulas of Olevskii [3] ($A = 24.0$; $B = 4.3$; $C_D = 0.3183$) and Heer and Fair [6] ($A = 24.0$; $B = 3$; $C_D = 0.34$) and many empirical formulas proposed by various authors are particular expressions of Eq. (7) [5, 6].

Moreover, it is easy to obtain Eq. (7) on the basis of dimensional analysis.

For a body of given shape the steady-state motion of the fluid is determined by a functional relation of the following form [7]:

$$F = f_1(d, v, \alpha, \rho, \mu). \tag{8}$$

Considering that for a sphere the function $f_2(\alpha) = \text{const} = c$, we obtain

$$F = cf(d,\ v,\ \rho,\ \mu). \tag{9}$$

Since the appearance of friction forces results in the formation of a boundary layer around the moving sphere, the reduced thickness of the boundary layer λ serves as a measure of these forces.

Then,

$$F = cf(d,\ v,\ \rho,\ \lambda). \tag{10}$$

Bearing in mind that d and λ have the same dimension, we can write

$$F = cf(D,\ v,\ \rho). \tag{11}$$

Thus, in solving Eq. (10) we can consider the motion of a sphere of somewhat greater diameter $D = d + 2\lambda$.

Applying the methods of dimensional analysis, we obtain

$$F = cD^a v^b \rho^c \tag{12}$$

or

$$MLT^{-2} = cL^a (LT^{-1})^b (ML^{-3})^c, \tag{13}$$

whence a = 2; b = 2; c = 1;

$$F = cD^2 v^2 \rho. \tag{14}$$

Representing Eq. (14) in dimensionless parameters, we obtain relation (7).

To solve Eq. (7) it is sufficient to know three values of Re and the corresponding values of ξ. On the basis of the carefully measured sphere drags reported in [8], the constant coefficients prove to be equal to A = 22.8; B = 6.67; $C_D = 0.333$.

The difference between the constant coefficients and those obtained by Stokes, Prandtl, and Newton is a consequence of the assumptions which they made in their theoretical solutions (according to Stokes A = 24.0 [4]; according to Prandtl B = 5.6 [9]; according to Newton $C_D = 0.5$ [5]). Moreover, in reality, over the entire range of Reynolds numbers all three types of drag are encountered, and considering one of them in isolation leads to an inaccurate result.

The calculated values from Eq. (7) and experimental data [1–3, 7, 8] are in good agreement (Figs. 1 and 2) up to $\mathrm{Re} = 3 \cdot 10^2$. In the region $\mathrm{Re} = 3 \cdot 10^2–10^4$, the calculated values of the drag coefficient ξ are somewhat higher, and in the region $\mathrm{Re} > 10^4$, they are somewhat lower than the experimental values.

To establish values of the drag coefficient in the region $\mathrm{Re} = 3 \cdot 10^2–10^4$, the author made multiple measurements of the uniform rate of fall of plexiglass spheres ($\gamma = 1.187$) 0.3–2.0 cm in diameter in water. The experiments were conducted in a vessel 160 cm tall measuring 20 × 20 cm in cross section at a water temperature of 20°. The time taken by the spheres to fall was measured with a stopwatch correct to 0.1 sec. Although the accuracy of such experiments is not high, they distinctly indicate higher values of the drag coefficient in the region $\mathrm{Re} = 3 \cdot 10^2–10^4$ as compared with the previously published data (Fig. 1). This suggests that Eq. (7) can be used for the motion of a sphere at Reynolds numbers up to 10^4 with an accuracy sufficient for practical purposes. The analogous three-term empirical formulas of Olevskii [3] and Fair and Heer [6] have the same limits of applicability. Thus, the assumptions made in deriving Eq. (7) are quite justified.

Representing Eq. (7) in dimensional form, we obtain

$$F = 2.85\,\pi\mu\, dv + 0.333\,\frac{\pi d^2}{4}\,\frac{\rho v^2}{2} + C_D'\pi d\,\lambda\,\frac{\rho v^2}{2}. \tag{15}$$

Equation (15) is an equation of the fourth degree in the velocity v. Therefore, its solution, which is very

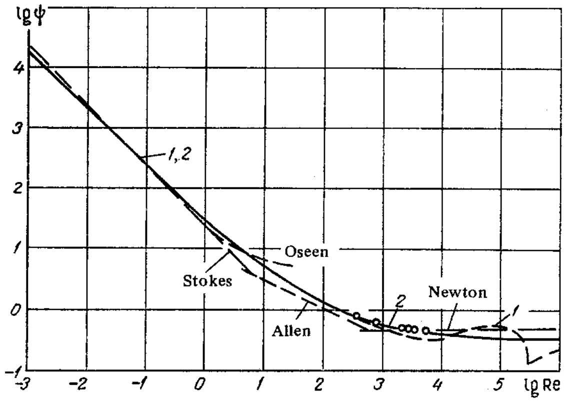

Fig. 1. Calculated and experimental $\psi = f(\mathrm{Re})$ curves: 1) experiment [1–3, 7, 8]; 2) calculated from Eqs. (7) and (19); the points represent the author's experiments.

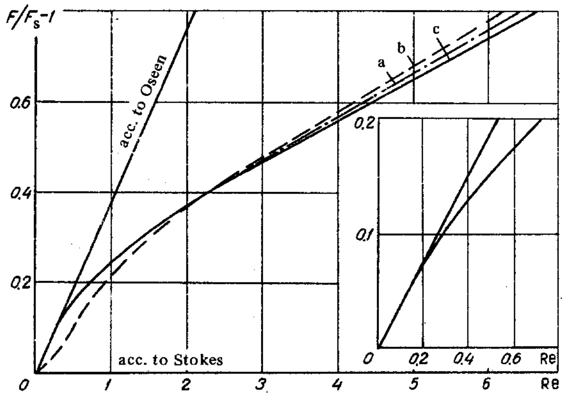

Fig. 2. Calculated and experimental $F/F_S - 1 = f(\mathrm{Re})$ curves (F—sphere drag; F_S—Stokes drag): a) Maxworthy's data [8]; b) Perry's data [8]; c) calculated from Eqs. (7) and (19).

frequently required in practice, is clumsy and inconvenient for practical calculations. To obtain more convenient expressions for the velocity of the sphere $v = f(d, \mu, \rho)$, we may approximately assume that λ is constant. Actually, considering the relation for λ and the fact that in the transition region, where it is impossible to neglect the last term in expression (15), the velocity is an almost linear function of the diameter, i.e., $v = k_A d$, we obtain

$$\lambda = k_1 k_2 \Big/ \sqrt{\frac{k_A \rho}{\mu}} = \text{const.}$$

To check the validity of this assumption, we use Eq. (7) to determine the fraction of the total drag represented by the term containing the quantity λ at various values of the Reynolds number:

$$F_\lambda = \frac{6.67}{\sqrt{\mathrm{Re}}\,\psi}. \tag{16}$$

We then determine the value of the error $\Delta\lambda$ (%) due to the fact that in reality λ does not remain constant:

$$\Delta\lambda = \frac{\left(k_1 k_2 d/\sqrt{\mathrm{Re}} - k_1 k_2 \Big/ \sqrt{\frac{k_A \rho}{\mu}}\right)}{k_1 k_2 d/\sqrt{\mathrm{Re}}}\, 100 \doteq \left(1 - \sqrt{\frac{v}{k_A d}}\right) 100. \tag{17}$$

Then the error ΔF (%) in calculating the drag at constant λ

$$\Delta F = F_\lambda \Delta\lambda. \tag{18}$$

Using Lashchenko's method [2, 3], for a given $\psi = f(\mathrm{Re})$ we can determine the values of v, d, and $k_A = \sqrt[3]{\left(\frac{4}{3}\frac{g}{\psi}\right)^2 \frac{1}{\mathrm{Re}}}\ \sqrt[3]{\frac{\Delta\rho^2}{\rho\mu}}$, assuming that $\mu = 0.01$ poise, $\rho = 1.0$ g/cm³, $\rho_1 = 7.8$ g/cm³, and $g = 981$ cm/sec². The calculations are summarized in the table, from which it is clear that in reality in the transition region (Re = 10–1000) the value of the coefficient k_A remains almost constant and, on the average, equal to 338, while the error in calculating the drag does not exceed 5%.

Absolute Errors in Calculating the Drag of a Rigid Sphere in a Viscous Medium at λ = const and $k_A = 338$

Re	ψ	v, cm/sec	d, cm	k_A	$\Delta\lambda$, %	F_λ	ΔF, %
0.001	23060	0.01570	0.0006370	24.40	73.1	0.0091	0.66
0.01	2352	0.07236	0.001376	52.50	60.5	0.0284	1.72
0.1	249.9	0.3300	0.003025	109.0	43.1	0.0845	3.65
1.0	29.85	1.440	0.06950	207.0	23.2	0.223	4.85
10.0	4.725	5.727	0.01746	328.0	1.50	0.445	0.67
100	1.228	19.35	0.05168	374.0	5.00	0.543	0.71
200	0.918	26.87	0.07442	361.0	3.00	0.490	1.47
1000	0.564	54.01	0.1850	292.0	7.00	0.372	2.60
4000	0.441	93.15	0.4295	217.0	20.0	0.238	4.75
10000	0.402	130.2	0.7680	169.5	29.2	0.166	4.85
20000	0.382	167.0	1.199	139.2	35.5	0.123	4.36
40000	0.380	210.6	1.990	111.0	45.3	0.0877	3.97

Since for a given $\psi = f(\mathrm{Re})$ the values of ψ and Re corresponding to the minimum error ΔF do not depend on the physical properties of the medium and the sphere, we can write $k_A = 20.29 \sqrt[3]{\frac{\Delta\rho^2}{\rho\mu}}$.

Considering that $B = 4C'_D k_1 k_2 = 6.67$; $k_A = 338$ (at $\mu = 0.01$ poise, $\rho = 1.0$ g/cm³, and $\rho_1 = 7.8$ g/cm³), the constant coefficient in the last term of Eq. (15) is equal to $K = C'_D k_1 k_2 \Big/ \sqrt{\frac{k_A \rho}{\mu}} = 0.009072$.

Then Eq. (15) finally reduces to the form

$$F = 2.85\,\pi\mu\, dv + K\,\pi\, d\,\frac{\rho\, v^2}{2} + 0.333\,\frac{\pi\, d^2}{4}\,\frac{\rho\, v^2}{2}. \tag{19}$$

Equating the drag force (19) to the difference between the gravitational and Archimedean forces, we obtain

$$v = -\frac{2.85\mu}{(K + d/12)\,\rho} + \sqrt{\left[\frac{2.85\mu}{\rho\,(K + d/12)}\right]^2 + \frac{d^2(\rho_1 - \rho)\,g}{3\,(K + d/12)\,\rho}}. \tag{20}$$

Equations (19) and (20) have the same limits of applicability as Eq. (7) ($Re \leq 10^4$).

NOTATION

F is the sphere drag force; C_V is the viscous drag coefficient; μ is the dynamic viscosity; d is the diameter of the sphere; v is the steady-state velocity of the sphere; C_D is the dynamic drag coefficient divided by the area of the maximum cross section of the moving sphere; ρ is the density of medium; r_0 is the radius of sphere; v_x, v_y, and v_z are the velocity components along the coordinate axes; Re is the dimensionless Reynolds number $\delta = k_2 d/\sqrt{Re}$ is the thickness of the boundary layer (Prandtl layer); $\lambda = k_1\delta$ is the reduced thickness of the boundary layer; k_2 is the dimensionless proportionality factor; k_1 is the dimensionless reduction coefficient; $S' = S + S_1$ is the total area; $S = \pi d^2/4$ is the area is the maximum cross section of the sphere; $S_1 = \pi(d\lambda + \lambda^2)$ is the reduced area of the boundary layer in the plane of the maximum cross section of the sphere perpendicular to the direction of motion; C'_D is the dynamic drag coefficient divided by the area S_1; $\psi = (4/3)(d(\rho_1 - \rho)g/\rho v^2)$ is the dimensionless drag coefficient; α is the angle of attack; $D = d + 2\lambda$ is the reduced diameter of the sphere; ρ_1 is the density of sphere; γ is the specific weight; K, k_A, A, B, c are coefficients which are constant for given conditions; g is the free-fall acceleration.

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